THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR ..MARIA MARGARET KLAWE..............................

TITLE OF THESIS ..STUDIES IN AMENABLE SEMIGROUPS....................

....

....

DEGREE FOR WHICH THESIS WAS PRESENTED ..PH. D......................

YEAR THIS DEGREE GRANTED ..1977......................................

THE UNIVERSITY OF ALBERTA

STUDIES IN AMENABLE SEMIGROUPS

by

© MARIA MARGARET KLAWE

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL 1977

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ....STUDIES IN AMENABLE SEMIGROUPS............... .... submitted by ....MARIA MARGARET KLAWE..................... in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

DEDICATION

For those rare moments of clarity,

When the dazzling intricacies of the patterns under the water

Are perceived,

If but for an instant.

ABSTRACT

Let S be a discrete semigroup. A left invariant mean on the Banach space $m(S)$ of bounded real-valued functions on S with the sup norm, is a positive element of norm one in the dual $m(S)^*$ of $m(S)$, which is invariant with respect to all left translations on $m(S)$ by elements of S. When $m(S)$ has a left invariant mean, we say that S is left amenable. In this thesis we present two sets of results on left amenable semigroups.

The first set, contained in Chapter II, is concerned with determining a lower bound for, and in some cases exactly, the dimension of the set of left invariant means (denoted by $\dim \langle M\ell(S)\rangle$) when S is left amenable. Theorem: If S is left amenable, then $\dim \langle M\ell(S)\rangle = n < \infty$ if and only if S contains exactly n disjoint finite left ideal groups. This result was proved by Granirer for S countable or left cancellative. Moreover, when S is infinite, left amenable, and either left or right cancellative, we show that $\dim \langle M\ell(S)\rangle$ is at least the cardinality of S. An application of these results shows that the radical of the second conjugate algebra of $\ell_1(S)$ is infinite dimensional when S is a left amenable semigroup which does not contain a finite ideal.

The purpose of the second set of results, comprising Chapter III, is to settle two problems. The first is Sorenson's conjecture on whether every right cancellative left amenable semigroup is left cancellative. The second, posed by Argabright and Wilde, is whether every left amenable semigroup satisfies the strong Følner condition.

We first show that these two problems are equivalent, then prove that the answer to both questions is no, through analyzing the semi-direct product of semigroups in relation to amenability and cancellation properties. Various other properties of semigroups satisfying the strong Følner condition and a subclass of such semigroups (called left measurable semigroups) are also discussed.

ACKNOWLEDGEMENT

I wish to express my deep gratitude to my supervisor, Anthony Lau. I am well aware of the sizeable amount of time that he has invested in directing my research. His comments and suggestions have often led to clearer proofs or further applications of my results. However, most of all, I would like him to know how much I value the interest, enthusiasm, and confidence that he has shown in my work.

I would also like to thank my family, especially my parents, who have helped me in so many ways.

Finally, my thanks go to the National Research Council of Canada, the University of Alberta, and the University of British Columbia, for all of their financial support.

TABLE OF CONTENTS

CHAPTER I

PRELIMINARIES

I.1. Introduction

Let S be a discrete semigroup, and $m(S)$ the Banach space of bounded real-valued functions with the sup norm. For each $s \in S$ we define a linear operator $\ell_s[r_s]$ on $m(S)$ by $\ell_s f(t) = f(st)$ $[r_s f(t) = f(ts)]$ for $t \in S$ and $f \in m(S)$. A <u>mean</u> on $m(S)$ is a positive element of norm one in the dual $m(S)^*$ of $m(S)$. We say that $\mu \in m(S)^*$ is <u>left</u> [<u>right</u>] <u>invariant</u> if $\mu(\ell_s(f)) = \mu(f)$ $[\mu(r_s(f)) = \mu(f)]$ for each $f \in m(S)$ and $s \in S$. Let $M\ell(S)$ $[Mr(S)]$ denote the set of left [right] invariant means on $m(S)$, and $\langle M\ell(S)\rangle$ $[\langle Mr(S)\rangle]$ denote its linear span in $m(S)^*$. When there exists a left [right] invariant mean on $m(S)$, we say that S is <u>left</u> [<u>right</u>] <u>amenable</u>. Furthermore, when S is both left and right amenable, we say that S is <u>amenable</u>.

This subject originates from a result of Banach in 1923 [2], who showed that there exists a mean on the bounded real-valued functions on the integers which is invariant under all translations, i.e. the group of integers under addition is amenable. This contrasted the result of Hausdorff in 1914 [20], who showed that there does not exist any mean on the bounded real-valued functions on the sphere in three dimensions, which is invariant under all rotations. In 1929 von Neumann [30] studied invariant means on $m(S)$ for S a group, and proved several basic combinatorial results about the class of

amenable groups. Extensions of some of these results to left amenable semigroups were obtained by Day [7] and Dixmier [9]. The first comprehensive discussion of the properties of left amenable semigroups was given by Day in 1957 [7], who also gave a survey of further results in this area in 1969 [8]. Two other general references on amenability are Hewitt and Ross [21, Section 17] and Greenleaf [19].

This thesis is basically composed of two sets of results on amenable semigroups. The first set, contained in Chapter II, is concerned with determining a lower bound for, and in some cases determining exactly, the dimension of $\langle M\ell(S)\rangle$ when S is left amenable. The purpose of the second set of results which comprises Chapter III, is to settle two problems; the first being Sorenson's conjecture on whether every right cancellative left amenable semigroup is left cancellative, and the second, the question posed by Argabright and Wilde of whether every left amenable semigroup must satisfy the strong Følner condition (SFC). We are able to show that the answer to both of these questions is no.

The thesis is organized as follows:

Chapter I contains some basic definitions and results that will be required.

In Chapter II we begin by describing previous results on the dimension of $\langle M\ell(S)\rangle$ (denoted by dim $\langle M\ell(S)\rangle$) obtained by Day, Luthar, Granirer, and Chou. In Sections II.2 and II.3 we investigate the structure of left thick subsets of left amenable semigroups. This technique leads to the proof of the main theorem (2.4.4) of this chapter, which extends the same result proved by Granirer in 1963 [14] for countable or left cancellative semigroups. Theorem 2.4.4:

If S is left amenable, then $\dim \langle M\ell(S)\rangle = n < \infty$ if and only if S contains exactly n disjoint finite groups which are left ideals. Moreover, we are able to show that when S is infinite, left amenable, and either right cancellative or left cancellative, then the dimension of $\langle M\ell(S)\rangle$ is at least the cardinality of S. An application of these results shows that the radical of the second conjugate algebra of $\ell_1(S)$ is infinite dimensional when S is a left amenable semigroup which does not contain a finite ideal (Theorem 2.5.1). After exhibiting various examples in Section II.7, Chapter II concludes with some comments on the limitations of using this type of method to determine the exact size of the dimension of $\langle M\ell(S)\rangle$.

The introduction in Chapter III gives the history of Sorenson's conjecture, and Argabright and Wilde's question of whether all left amenable semigroups satisfy SFC. We then show in Section III.2 that these two problems are actually equivalent; this result follows directly from Theorem 3.2.2 which completely characterizes the semigroups which satisfy SFC, as those left amenable semigroups whose right cancellative quotient semigroups are left cancellative. Then in Section III.3, we prove that the answer to both questions is no, through analyzing the semidirect product of semigroups in relation to amenability and cancellation properties. In Section III.4 we investigate some of the properties of semigroups satisfying SFC, and in Section III.5 we include some analogous results obtained by Sorenson for left measurable semigroups.

I.2 General properties of left amenable semigroups

For any subset A of a semigroup S, and $s \in S$, we define the sets $sA = \{st : t \in A\}$, $As = \{ts : t \in A\}$, $s^{-1}A = \{t \in S : st \in A\}$, and $As^{-1} = \{t \in S : ts \in A\}$. The cardinality of A is denoted by $|A|$, and its characteristic function by χ_A, i.e. $\chi_A(s) = 1$ if $s \in A$ and $\chi_A(s) = 0$ if $s \notin A$.

Notice that if S is left amenable, then for any $\mu \in M\ell(S)$, $A \subset S$, and $s \in S$ we have $\mu(\chi_{s^{-1}A}) = \mu(\chi_A)$ since $\ell_s\chi_A = \chi_{s^{-1}A}$. Thus for each $s \in S$ we have $\mu(\chi_{sS}) = 1$ since $1 = \mu(\chi_S) \geq \mu(\chi_{sS}) = \mu(\chi_{s^{-1}(sS)}) = \mu(\chi_S) = 1$. From this, it is easy to see that in a left amenable semigroup, the intersection of finitely many right ideals is always non-empty.

The class of left amenable semigroups includes all finite groups, solvable groups, and abelian semigroups (for proofs see Greenleaf [19, Chapter 1] or Hewitt and Ross [21, Section 17]). As examples of semigroups which are not left amenable, consider a free semigroup on more than one generator, or any semigroup S with $|S| > 1$ and the multiplication $st = s$ for all $s,t \in S$. Since both of these semigroups contain disjoint right ideals, they are not left amenable. Von Neumann showed that any free group on more than one generator is not left amenable; for a proof of this see Greenleaf [19, Ex. 1.2.3] or Hewitt and Ross [21, 17.16]. Undoubtedly the most fundamental open problem remaining in the study of amenable groups, is the question of whether every group which is not amenable contains a free subgroup on two generators (see Day [8, p. 12]).

Homomorphic images and directed unions of left amenable semigroups are left amenable. Also, any subgroup of a left amenable group is left amenable. However, a subsemigroup of a left amenable group need not be left amenable (see Hochster [22]).

I.3 The right cancellative quotient semigroup

A semigroup S is said to be <u>right</u> [<u>left</u>] <u>cancellative</u> if whenever $rs = ts$ $[sr = st]$ we have $r = t$. We define a relation R on any semigroup S by sRt for $s,t \in S$ if there exists $x \in S$ with $sx = tx$. If the intersection of finitely many right ideals of S is always non-empty (as is the case when S is left amenable), then R is an equivalence relation, the set S' of of equivalence classes is a right cancellative semigroup under the induced multiplication, and the quotient map $\pi : S \rightarrow S'$ is a semigroup homomorphism. A detailed discussion on this relation R is found in Granirer [15, p. 371]. When S' exists, we will refer to it as the <u>right cancellative quotient semigroup</u> of S. Notice that if S is left amenable, then by Proposition 1.2.1 the semigroup S' is also left amenable.

I.4 Left thickness and left amenability

Let A and B be subsets of a semigroup S. Then A is said to be left thick in B if for every finite subset $F \subset B$, there exists $s \in S$ such that $Fs \subset A$. Notice that left thickness is a transitive property, i.e. if A is left thick in B, and B is left thick in C, then A is left thick in C. When a subset is left thick in the semigroup itself, then we refer to it as a left thick subset of the semigroup. Clearly any left ideal of a semigroup is a left thick subset.

The definition of left thickness is due to Mitchell, who gave the following characterization of the left thick subsets of left amenable semigroups in [25, Thm. 7].

Theorem (Mitchell): If S is a left amenable semigroup, then a subset A is left thick in S if and only if there exists $\mu \in M\ell(S)$ with $\mu(\chi_A) = 1$.

Notice that this shows that every right ideal of a left amenable semigroup is a left thick subset.

CHAPTER II

ON THE DIMENSION OF LEFT INVARIANT MEANS AND LEFT THICK SUBSETS

II.1 Introduction

It is natural to look for conditions which determine the dimension of $\langle M\ell(S)\rangle$ when S is left amenable. The first results in this direction were given by Day [7, p. 535], who proved that infinite solvable groups, infinite amenable non-torsion groups, and infinite locally finite groups, all have more than one left invariant mean. In [24, p. 43], Luthar showed that a commutative semigroup has a unique left invariant mean if and only if it has a finite ideal. Continuing the search for conditions that $\langle M\ell(S)\rangle$ be finite dimensional, Granirer [14, p. 32] proved the following theorem:

THEOREM (Granirer): If S is a countably infinite left amenable semigroup, then $\dim \langle M\ell(S)\rangle = n < \infty$ if and only if S contains exactly n disjoint finite groups which are left ideals.[1]

Except for the case where S is left cancellative [14, p. 49], Granirer was unable to drop the countability condition; he was only able to replace it by a slightly weaker one (see [14, p.44]).

When S is an infinite left amenable semigroup with cancellation, Chou has proved [4] that $\dim \langle M\ell(S)\rangle \geq 2^{\mathfrak{c}} \cdot |S|$, where $\mathfrak{c}$ denotes the cardinality of the continuum, and $|S|$ the cardinality of S. Later Chou [5], using an idea of Kakutani and Oxtoby in [21, §16], was able to prove that $\dim \langle M\ell(S)\rangle = 2^{2^{|S|}}$ when S is an infinite amenable group.

In this chapter we investigate the structure of the left thick subsets of infinite left amenable semigroups, and use this to obtain lower bounds for the dimension of the set of left invariant means. By this method, we are able to prove Granirer's theorem without the countability condition, and hence prove Luthar's result for all left amenable semigroups. Another easily obtained corollary is that if S is amenable (both left and right) and $\dim \langle M\ell(S)\rangle$ is finite, then $\dim \langle M\ell(S)\rangle = \dim \langle Mr(S)\rangle = 1$ (i.e. S has a unique invariant mean). This generalizes another theorem of Granirer [15, Thm. 1] who proved the above proposition under the additional hypothesis that $\dim \langle Mr(S)\rangle$ be finite. Our techniques also yield that if S is either right cancellative or left cancellative as well as infinite and left amenable, then $\dim \langle M\ell(S)\rangle \geq |S|$.

In Section II.2 we begin by noting (Remark 2.2.1) the relationship between collections of pairwise disjoint left thick subsets of a semigroup S and $\dim \langle M\ell(S)\rangle$. We then introduce the concept of strong left thickness for subsets of S . Theorem 2.2.2 illustrates how this concept is related to obtaining "large" collections of pairwise disjoint left thick subsets in S . This section concludes with some specific results for right cancellative semigroups.

In Section II.3 we define uniform left thickness, and give an equivalent characterization in terms of the behaviour of left invariant means on the characteristic functions of subsets of S with smaller cardinality than S (Proposition 2.3.1).

Our main result giving a lower bound for $\dim \langle M\ell(S)\rangle$ is found in Section II.4 (Theorem 2.4.1) as well as the generalizations of the

theorems of Granirer and Luthar (Theorem 2.4.4, Corollaries 2.4.5 and 2.4.6).

Section II.5 gives an application of the results of Section II.4 to the radical of the second conjugate algebra $m(S)^*$. In Section II.6 we look at thickness properties for left cancellative semigroups, while Section II.7 is devoted to examples. Section II.8 considers the limitations involved in using our techniques to find the exact size of $M\ell(S)$.

II.2 Strong left thickness and collections of pairwise disjoint left thick subsets

REMARK 2.2.1. If $\{A_\gamma : \gamma \in \Gamma\}$ is a collection of pairwise disjoint left thick subsets of a left amenable semigroup S, then by Mitchell's theorem (see Section I.4) for each $\gamma \in \Gamma$ we can choose $\mu_\gamma \in M\ell(S)$ with $\mu_\gamma(\chi_{A_\gamma}) = 1$. This set $\{\mu_\gamma : \gamma \in \Gamma\}$ of left invariant means on S is linearly independent. In fact if $\sum_{i=1}^{n} a_i\mu_i = 0$ for some $\{\mu_1,\ldots,\mu_n\} \subset \{\mu_\gamma : \gamma \in \Gamma\}$, then for $j = 1,\ldots,n$ we have $a_j = \sum_{i=1}^{n} a_i\mu_i(\chi_{A_j}) = 0$. Thus we are interested in finding collections of pairwise disjoint left thick subsets of S.

We say that $A \subset S$ is <u>strongly left thick</u> if for each $B \subset S$ with $|B| < |A|$, the set $A \setminus B$ is left thick in S. Although every semigroup is obviously left thick in itself, it is easy to find examples of left amenable semigroups which are not strongly left thick. Consider finite groups for instance, or left amenable semigroups which contain a right ideal of smaller cardinality. An example of this last type is given in Section II.7 (Example 2.7.3).

The following theorem shows that strong left thickness is a useful property in determining lower bounds for the dimension of $\langle M\ell(S)\rangle$.

THEOREM 2.2.2. A subset $A \subset S$ is strongly left thick if and only if there exists a collection $\{D_\gamma : \gamma \in \Gamma\}$ of pairwise disjoint subsets of A, which are left thick in S, such that $|\Gamma| = |A|$.

Proof: Suppose A is strongly left thick. If A is finite, then for each $a \in A$ we have $\{a\} = A \setminus (A \setminus \{a\})$ is left thick in S; hence $\{\{a\} : a \in A\}$ is the desired collection. Thus we may assume A is infinite.

Let ω be the first ordinal with $|\omega| \equiv |\{\alpha : \alpha < \omega\}| = |A|$. Write $A = \{a_\alpha : \alpha < \omega\}$ and let $A_\beta = \{a_\alpha : \alpha < \beta\}$ if $1 \leq \beta < \omega$. Using transfinite induction we construct a family

$$\{t_{(\beta,F)} : 1 \leq \beta < \omega, \ F \text{ a finite subset of } A_\beta\}$$

with the following properties:

(i) $Ft_{(\beta,F)} \subset A$ for each finite $F \subset A_\beta$, where $1 \leq \beta < \omega$.

(ii) Let $X_\beta = \bigcup\{Ft_{(\beta,F)} : \text{finite } F \subset A_\beta\}$ if $1 \leq \beta < \omega$.

Then $X_\beta \cap X_\delta = \phi$ for $1 \leq \beta < \delta < \omega$.

$A_1 = \{a_o\}$ and since A is left thick, there exists $t \in S$ such that $a_o t \in A$. Let $t_{(1,\{a_o\})} = t$.

Suppose we have constructed a family $\{t_{(\beta,F)} : 1 \leq \beta < \delta$, F finite $\subset A_\beta\}$ satisfying (i) and (ii), where $1 < \delta < \omega$. Let $Y = \bigcup\{X_\beta : 1 \leq \beta < \delta\}$, and let $|\delta|$ denote $|\{\alpha : \alpha < \delta\}|$. If $|\delta|$ is finite, then Y is a finite union of finite sets so $|Y| < |A|$. If $|\delta|$ is infinite, then $|X_\beta| \leq |\delta|$ for $1 \leq \beta < \delta$, and we have $|Y| \leq |\delta|^2 = |\delta| < |A|$, since $\delta < \omega$. Thus $|Y| < |A|$ in either case, hence $A \setminus Y$ is left thick, and so for each finite subset $F \subset A_\delta$ we can choose $t_{(\delta,F)} \in S$ such that $Ft_{(\delta,F)} \subset A \setminus Y$. Then $X_\delta \subset A \setminus Y$ and $X_\beta \cap X_\delta = \phi$ for $1 \leq \beta < \delta$.

Let $\Gamma = \{\gamma : 1 \leq \gamma < \omega\}$. Choose a 1-1 correspondence $T : \Gamma \to \Gamma \times \Gamma$ and define $\Gamma_\gamma = T^{-1}(\Gamma \times \{\gamma\})$ for each $\gamma \in \Gamma$. Now each Γ_γ is cofinal in Γ since $|\Gamma_\gamma| = |\Gamma|$, and if $\gamma_1 \neq \gamma_2$ then $\Gamma_{\gamma_1} \cap \Gamma_{\gamma_2} = \phi$.

For each $\gamma \in \Gamma$, define $D_\gamma = \bigcup\{X_\beta : \beta \in \Gamma_\gamma\}$. Clearly $\{D_\gamma : \gamma \in \Gamma\}$ is a collection of pairwise disjoint subsets of A . Since $|\Gamma| = |\omega| = |A|$, it only remains to show that each D_γ is left thick in S . Since A is left thick in S, it suffices to show that D_γ is left thick in A . Let F be a finite subset of A . Then for some β, we must have $F \subset A_\beta$, and since Γ_γ is cofinal in Γ we may assume $\beta \in \Gamma_\gamma$. Now $Ft_{(\beta,F)} \subset X_\beta \subset D_\gamma$, and we see that D_γ is left thick.

Conversely, let $\{D_\gamma : \gamma \in \Gamma\}$ be a collection of pairwise disjoint subsets of A, which are left thick in S, such that $|\Gamma| = |A|$. Suppose $B \subset S$ with $|B| < |A|$. Since $|\Gamma| > |B|$, there exists $\gamma \in \Gamma$ with $B \cap D_\gamma = \phi$. Then $D_\gamma \subset A \setminus B$ so $A \setminus B$ is left thick, which finishes the proof.

Next we show that every infinite left amenable semigroup with right cancellation is strongly left thick. In Section II.6, we will see that this remains true when right cancellation is replaced by left cancellation. First we need the following lemmas:

LEMMA 2.2.3. If B and C are subsets of an infinite semigroup S with right cancellation such that $|B| < |S|$ and $|C| < |S|$, then there exists $s \in S$ with $B \cap sC = \phi$.

Proof. If not, then for each $s \in S$ we can find a pair $(b_s, c_s) \in B \times C$ such that $b_s = sc_s$. If $s \neq t$, then $(b_s, c_s) \neq (b_t, c_t)$ since otherwise we would have $sc_s = b_s = b_t = tc_t = tc_s$, and hence $s = t$ by right cancellation. Thus $|B \times C| = |B||C| \geq |S|$, which contradicts $|B| < |S|$ and $|C| < |S|$ since S is infinite.

LEMMA 2.2.4. If B is a subset of an infinite semigroup S with right cancellation such that $|B| < |S|$, then there exists a sequence $\{s_n\} \subset S$ with $s_nB \cap s_mB = \phi$ for $n \neq m$.

Proof. Construct the sequence $\{s_n\}$ by induction. Choose $s_1 \in S$ arbitrarily. If we have constructed $s_1, \ldots, s_n$ such that $s_iB \cap s_jB = \phi$ for $1 \leq i < j \leq n$, then by Lemma 2.2.3 we can find $s_{n+1} \in S$ so that $s_{n+1}B \cap (s_1B \cup \ldots \cup s_nB) = \phi$, since $|s_1B \cup \ldots \cup s_nB| < |S|$.

PROPOSITION 2.2.5. If S is an infinite left amenable semigroup with right cancellation, then for each $B \subset S$ with $|B| < |S|$ and for each $\mu \in M\ell(S)$ we have $\mu(\chi_B) = 0$.

Proof. By Lemma 2.2.4 there exists a sequence $\{s_n\}$ with $s_nB \cap s_mB = \phi$ for $n \neq m$. Thus $1 = \mu(\chi_S) \geq \sum_{n=1}^{\infty} \mu(\chi_{s_nB})$. For each n we have $\ell_{s_n} \chi_{s_nB} \geq \chi_B$ and so $\mu(\chi_{s_nB}) = \mu(\ell_{s_n} \chi_{s_nB}) \geq \mu(\chi_B)$. Hence $\mu(\chi_B) = 0$.

COROLLARY 2.2.6. If S is an infinite left amenable semigroup with right cancellation, then every left thick subset A is strongly

left thick.

Proof: Choose $\mu \in M\ell(S)$ so that $\mu(\chi_A) = 1$. If $B \subset S$ with $|B| < |A|$, then $\mu(\chi_{(A \setminus B)}) \geq \mu(\chi_A) - \mu(\chi_B) = 1$. Thus $A \setminus B$ is left thick.

II.3 Uniform left thickness

In this section we examine another thickness property for semigroups. We say S is <u>uniformly left thick</u> if $|A| = |S|$ for every left thick subset $A \subset S$. As a corollary to the next proposition, we see that for infinite left amenable semigroups, uniform left thickness implies strong left thickness.

PROPOSITION 2.3.1. An infinite left amenable semigroup S is uniformly left thick if and only if for each $B \subset S$ with $|B| < |S|$ and for each $\mu \in M\ell(S)$ we have $\mu(\chi_B) = 0$.

Proof. Suppose S is uniformly left thick, $B \subset S$ with $|B| < |S|$, and $\mu \in M\ell(S)$. Let S' be the right cancellative quotient semigroup of S, and $\pi : S \longrightarrow S'$ the quotient map (see Section I.3 for definition). If $|S'| = |S|$, define $\mu' \in m(S')^*$ by $\mu'(f) = \mu(f \circ \pi)$ for each $f \in m(S')$. It is not difficult to show that in fact we have $\mu' \in M\ell(S')$. Moreover $|\pi(B)| < |S'|$, so by Proposition 2.2.5 we have $0 = \mu'(\chi_{\pi(B)}) = \mu(\chi_{\pi(B)} \circ \pi) \geq \mu(\chi_B) \geq 0$.

Thus we may assume that $|S'| < |S|$. For each $g \in S'$ choose $t_g \in \pi^{-1}(g)$. Let $C = \{t_g b : g \in S', b \in B\}$. Since S is infinite, $|C| < |S|$. If $\mu(\chi_B) > 0$, then C is left thick in S. To see this, we first show that for any finite set $F \subset S$, there exists $u \in S$ such that $Fu = \{t_g u : g \in \pi(F)\}$. It is easily seen by induction on $|F|$, that for each $g \in \pi(F)$ there exists $u_g \in S$ with $\left(F \cap \pi^{-1}(g)\right)u_g = t_g u_g$. Now choose $u \in \bigcap_{g \in \pi(F)} u_g S$. Then $Fu = \{t_g u : g \in \pi(F)\}$. If $\mu(\chi_B) > 0$, then $B \cap uS \neq \phi$ since

$\mu(\chi_{uS}) = 1$. Therefore we can find $s \in S$ so that $us \in B$. But now $Fus = \{t_g us : g \in \pi(F)\} \subset C$. Hence C is left thick. Since S is uniformly left thick and $|C| < |S|$, we must have $\mu(\chi_B) = 0$.

Conversely, if for each $B \subset S$ with $|B| < |S|$ we have $\mu(\chi_B) = 0$ for all $\mu \in M\ell(S)$, then clearly B is not left thick in S .

COROLLARY 2.3.2. For infinite left amenable semigroups, uniform left thickness implies strong left thickness.

The proof is identical to that of Corollary 2.2.6.

COROLLARY 2.3.3. Every infinite left amenable semigroup with right cancellation is uniformly left thick.

This follows immediately from Proposition 2.2.5.

REMARK 2.3.4. The above proposition fails when we allow S to be finite. Suppose S is a finite group. Then its only left thick subset is the group itself; so S is uniformly left thick. However, if μ is the unique left invariant mean on S, then
$\mu(\chi_B) = |B||S|^{-1} \neq 0$ if $B \neq \phi$.

In Section II.7 we give an example of an infinite left amenable semigroup which is strongly left thick but not uniformly left thick. Since most of the proof of Proposition 2.3.1 is devoted to the case where $|S'| < |S|$, an example of a uniformly left thick infinite left amenable semigroup S with $|S'| < |S|$ is also given in Section II.7. In view of Proposition 2.3.1, it is interesting to note a similar reformulation of strong left thickness for left amenable semigroups.

PROPOSITION 2.3.5. A left amenable semigroup is strongly left thick if and only if for each $B \subset S$ with $|B| < |S|$ there exists $\mu \in M\ell(S)$ with $\mu(\chi_B) = 0$.

II.4 A lower bound for $\dim \langle M\ell(S)\rangle$

We define the <u>width of a semigroup S</u>, denoted by $W(S)$, as $W(S) = \sup\{|A| : A$ is a strongly left thick subset of $S\}$. Now we are ready to prove our main result.

THEOREM 2.4.1. If S is a left amenable semigroup which contains no finite left thick subsets, then $\dim \langle M\ell(S)\rangle \geq W(S) \geq \aleph_0$.

Proof: That $\dim \langle M\ell(S)\rangle \geq W(S)$ follows immediately from Theorem 2.2.2 and Remark 2.2.1. Thus we need only show that S contains an infinite strongly left thick subset. Let A be a left thick subsemigroup of S so that $|A|$ is minimal. Clearly A is infinite, and also uniformly thick since if $B \subset A$ is left thick in A with $|B| < |A|$, then the subsemigroup generated by B is left thick in S, and has cardinality less than A. Since A is left thick in S, the semigroup A is left amenable. Thus by Corollary 2.3.2, A is strongly left thick.

REMARK 2.4.2. Even when S contains no finite left thick subsets, we may have $W(S) < |S|$ (see example 2.7.3).

With the help of this theorem, we proceed to give a proof of Granirer's theorem without the countability condition, and then the generalization of Luthar's result, as mentioned in Section II.1. For this we need one more lemma.

We use the term <u>left ideal group</u> to signify a group which is also a left ideal in S.

LEMMA 2.4.3. If a left amenable semigroup S contains a finite

left thick subset, then S contains a finite left ideal group.

Proof: Suppose $A \subset S$ is a finite left thick subset. The left ideal Sa is finite for some $a \in A$, since $\mu(\chi_A) = 1$ for some $\mu \in M\ell(S)$, implies $\mu(\chi_{\{a\}}) > 0$ for some $a \in A$. Choose $t \in S$ such that St is finite, with $|St|$ minimal, and let $C = St$. It is easy to check that C is right cancellative, and since C is also finite and left amenable, C is a finite group.

THEOREM 2.4.4. A left amenable semigroup has $\dim \langle M\ell(S)\rangle = n < \infty$ if and only if S contains exactly n disjoint finite left ideal groups.

Proof: In [14, p. 34], Granirer proved that if S contains exactly n disjoint finite left ideal groups, then $\dim \langle M\ell(S)\rangle = n$. Now suppose $\dim \langle M\ell(S)\rangle = n < \infty$. The semigroup S must contain some finite left ideal group since otherwise Lemma 2.4.2 and Theorem 2.4.1 show that $\dim \langle M\ell(S)\rangle$ is infinite. S cannot contain infinitely many disjoint finite left ideal groups since by Remark 2.2.1, that would again imply that $\dim \langle M\ell(S)\rangle$ is infinite. Thus S contains exactly m disjoint finite left ideal groups for some finite number m, and by Granirer's result, we see $m = n$.

COROLLARY 2.4.5. If S is a left amenable semigroup, then $\dim \langle M\ell(S)\rangle$ is finite if and only if S contains a finite two-sided ideal.

Proof: If $\dim \langle M\ell(S)\rangle = n < \infty$, then S contains exactly n disjoint finite left ideal groups, say $A_1,\ldots,A_n$. In [14, p. 34],

Granirer shows that $\bigcup\{A_i : i = 1,\dots,n\}$ is a finite two-sided ideal.

If A is a finite two-sided ideal of S, then for each $\mu \in M\ell(S)$ we have $\mu(\chi_A) = 1$, and hence $\dim \langle M\ell(S)\rangle = \dim \langle M\ell(A)\rangle$, which is finite since A is finite.

COROLLARY 2.4.6. If S is an amenable (both left and right) semigroup with $\dim \langle M\ell(S)\rangle = n < \infty$, then S contains a finite group which is a two-sided ideal, and hence $\dim \langle M\ell(S)\rangle = \dim \langle Mr(S)\rangle = 1$ (i.e. S has a unique invariant mean).

Proof: As shown in Corollary 2.4.5, S contains a finite two-sided ideal A which is the union of n disjoint finite left ideal groups. By the footnote 1 (p. 52) we see that A is left cancellative. Since A is also right amenable and finite, A is a finite group. Let $\mu \in m(S)^*$ be defined as $\mu(f) = |A|^{-1} \Sigma\{f(a) : a \in A\}$. Then $M\ell(S) = Mr(S) = \{\mu\}$.

Theorem 2.4.4 is proved by Granirer for S countable [14, p. 32], or S left cancellative [14, p. 49]. Moreover, he proved Corollary 2.4.6 for these cases [14, p. 46], and also for the general case under the additional assumption that $\dim \langle Mr(S)\rangle$ be finite [15, Thm. 1]. Corollary 2.4.5 was proved by Luthar [24, p.43] for S commutative.

II.5 An application to the radical of the second conjugate algebra $m(S)^*$

Let $\ell_1(S)$ be the space of real-valued functions θ on S such that $\Sigma\{|\theta(s)| : s \in S\}$ is finite. For $\nu \in m(S)^*$ and $f \in m(S)$, we define $\nu * f \in m(S)$ by $\nu * f(s) = \nu(\ell_s f)$. Now for $\mu,\nu \in m(S)^*$, the Arens multiplication on $m(S)^*$ is defined as $\mu * \nu(f) = \mu(\nu * f)$ for each $f \in m(S)$. Under this multiplication $m(S)^*$ becomes the second conjugate algebra of $\ell_1(S)$ (for more details see Day [7, p.526]).

THEOREM 2.5.1. If S is a left amenable semigroup which contains no finite ideals, then the radical J of $m(S)^*$ is infinite dimensional.

Proof: Using an ideal of Civin and Yood (see [6, pp. 849-850]), it can be shown that $M\ell(S) \subset J + \mu$ for any $\mu \in M\ell(S)$. Since $M\ell(S)$ is infinite dimensional by Corollary 2.4.5, the radical J must be also.

Granirer used the same proof to show that the radical of $m(S)^*$ is infinite dimensional when S is a commutative semigroup without finite ideals (see [15, p. 378]), and also when S is an infinite left amenable group (see [14, p. 48]).

COROLLARY 2.5.2. Let S be a left amenable semigroup and suppose the radical of $m(S)^*$ is finite dimensional. If S is either left cancellative or right cancellative, then S is finite.

Proof: By Theorem 2.5.1, S contains a finite ideal A . Now for any $a \in A$, we have $aS \cup Sa \subset A$. If S is either left cancellative or right cancellative, then $|S| \leq |aS \cup Sa| \leq |A|$, and we see that S is finite.

REMARK. This result is already known for the left cancellative case (Granirer and Rajagopalan [18]). It is unknown whether it still holds when the cancellative properties of S are dropped. However, the following example shows that the condition that S be left amenable and contain a finite ideal is not enough to ensure that the radical of $m(S)^*$ be finite dimensional.

EXAMPLE 2.5.3. Let $S = \{s_n : n = 0,1,2,...\}$ with the multiplication $s_i s_j = s_o$ for all i,j. Then $\mu \in m(S)^*$ defined by $\mu(f) = f(s_o)$ for each $f \in m(S)$, is the unique left (and right) invariant mean on S, and $\{s_o\}$ is a finite ideal in S. For each $n \geq 1$, define $\phi_n \in m(S)^*$ by $\phi_n(f) = f(s_{n+1}) - f(s_n)$ for each $f \in m(S)$. Now the set $\{\phi_n : n = 1,2,...\}$ is linearly independent in $m(S)^*$, and for any $\nu \in m(S)^*$ we have $\nu * \phi_n = \phi_n * \nu = 0$ for each n. Thus $\{\phi_n : n = 1,2,...\} \subset J$, and hence J is infinite dimensional.

A further illustration that the radical J of $m(S)^*$ may be much larger than the ideal $A = \{\mu_1 - \mu_2 : \mu_1,\mu_2 \in M\ell(S)\}$ is given by Civin and Yood in [6, Theorem 3.5], where they show that J/A is infinite dimensional when S is the additive group of integers.

II.6 Thickness properties for infinite left amenable semigroups with left cancellation

The purpose of this section is to show that every infinite left amenable semigroup with left cancellation is strongly left thick. In order to do this, we first prove some preliminary lemmas.

LEMMA 2.6.1. If B and C are subsets of an infinite semigroup S with left cancellation, such that $|B| < |S|$ and $|C| < |S|$, then there exists $s \in S$ with $B \cap Cs = \phi$.

Proof: Replace right cancellation by left cancellation in the proof of Lemma 2.2.3.

Given $s \in S$ and $\mu \in M\ell(S)$, define $\mu_{(s)} \in m(S)^*$ by $\mu_{(s)}(f) = \mu(r_s f)$ for each $f \in m(S)$, where $r_s f(t) = f(ts)$ for $t \in S$. It is easily checked that $\mu_{(s)} \in M\ell(S)$, and that for each $A \subset S$, we have $\mu_{(s)}(\chi_{As}) \geq \mu(\chi_A)$.

LEMMA 2.6.2. If S is an infinite left amenable semigroup with left cancellation and $B \subset S$ with $|B| < |S|$, then there exists a sequence $\{\mu_n\} \subset M\ell(S)$ such that $\sum_{n=1}^{\infty} \mu_n(\chi_B) \leq 1$.

Proof: Using induction, we construct sequences $\{\mu_1, \mu_2, \ldots\} \subset M\ell(S)$ and $\{s_2, s_3, \ldots\} \subset S$ with the following property:

(i) $$\mu_n\left(\chi_{(B \cup Bs_n \cup Bs_{n-1}s_n \cup \ldots \cup Bs_2 s_3 \ldots s_n)}\right) \geq \mu_n(\chi_B) + \mu_{n-1}(\chi_B) + \ldots + \mu_1(\chi_B)$$

Clearly this sequence $\{\mu_n\}$ satisfies the statement of the lemma, since

(i) implies that $1 = \mu_n(\chi_S) \geq \mu_n(\chi_B) + \mu_{n-1}(\chi_B) + \ldots + \mu_1(\chi_B)$ for each n. Choose $\mu_1 \in M\ell(S)$ arbitrarily. By Lemma 2.6.1 we can find $s_2 \in S$ with $B \cap Bs_2 = \phi$. Let $\mu_2 = \mu_{1(s_2)}$ as defined above.

If we have constructed $\{\mu_1,\ldots,\mu_{n-1}\}$ and $\{s_2,\ldots,s_{n-1}\}$ for some $n \geq 3$, then by Lemma 2.6.1 we can find s_n with $B \cap (B \cup Bs_n \cup Bs_{n-1} \cup \ldots \cup Bs_2s_3\ldots s_{n-1})s_n = \phi$. Let $\mu_n = \mu_{n-1(s_n)}$. Then we have

$$\mu_n\left(\chi_{(B \cup Bs_n \cup Bs_{n-1}s_n \cup \ldots \cup Bs_2s_3\ldots s_n)}\right)$$

$$= \mu_n(\chi_B) + \mu_n\left(\chi_{(B \cup Bs_{n-1} \cup Bs_{n-2}s_{n-1} \cup \ldots \cup Bs_2s_3\ldots s_{n-1})s_n}\right)$$

$$\geq \mu_n(\chi_B) + \mu_{n-1}\left(\chi_{(B \cup Bs_{n-1} \cup Bs_{n-2}s_{n-1} \cup \ldots \cup Bs_2s_3\ldots s_{n-1})}\right)$$

$$\geq \mu_n(\chi_B) + \mu_{n-1}(\chi_B) + \mu_{n-2}(\chi_B) + \ldots + \mu_1(\chi_B) .$$

PROPOSITION 2.6.3. If S is an infinite left amenable semigroup with left cancellation, then S is strongly left thick.

Proof: Let $B \subset S$ with $|B| < |S|$. By Lemma 2.6.2 we can find a sequence $\{\mu_n\} \subset M\ell(S)$ with $\sum_{n=1}^{\infty} \mu_n(\chi_B) \leq 1$. Clearly we must have $\lim_{n\to\infty} \mu_n(\chi_B) = 0$. Since $M\ell(S)$ is weak*-compact, we can find a subnet (μ_α) of $\{\mu_n\}$ which is weak*-convergent to $\mu \in M\ell(S)$. Then $\mu(\chi_B) = 0$, so $\mu(\chi_{(S \setminus B)}) = 1$ and hence $S \setminus B$ is left thick in S.

REMARK. Although every infinite left amenable semigroup with right cancellation is uniformly left thick, this does not remain true when right cancellation is replaced by left cancellation (see Example 2.7.4).

II.7 Examples

The first three examples are special cases of the following general type of left amenable semigroup. If S is a lattice (a partially ordered set in which every finite set has a supremum and an infimum), then a multiplication can be defined on S by $ab = \sup\{a,b\}$ for $a,b \in S$. This operation is commutative and associative and hence S is a left amenable semigroup (S is even extremely left amenable, i.e. $m(S)$ admits a multiplicative left invariant mean, see [16]). We call this operation the sup multiplication. A subset $A \subset S$ is left thick in S if and only if A is cofinal in S .

EXAMPLE 2.7.1. A left amenable semigroup containing no finite left thick subsets, which is strongly left thick but not uniformly left thick.

Let S be the real numbers under the sup multiplication. The set of integers is left thick in S; hence S is not uniformly left thick. If $B \subset S$ with $S \setminus B$ not left thick, then the interval $[a,\infty)$ is contained in B for some $a \in S$, and we have $|B| = |S|$. Thus S is strongly left thick.

EXAMPLE 2.7.2. An infinite uniformly left thick left amenable semigroup S which is extremely left amenable. In particular $|S'| < |S|$.

Let S be the integers under the sup multiplication. Obviously S is uniformly left thick since every cofinal subset is infinite. Also $|S'| = 1$ since for each $a,b \in S$ there exists $c \in S$ with $ac = bc$.

EXAMPLE 2.7.3. A left amenable semigroup containing no finite left thick subsets, which is not strongly left thick.

Let S be the union of the interval $[0,1]$ with the natural numbers $\mathbb{N}$ under the sup multiplication. S has no finite left thick subsets since every cofinal subset is infinite. Since $|\mathbb{N}| < |S|$, and $S \setminus \mathbb{N}$ is not left thick, S is not strongly left thick.

Since $\mathbb{N}$ is a right ideal of S, it is natural to ask whether every infinite left amenable semigroup, which is not strongly left thick, contains a right ideal of smaller cardinality. As far as we know, this question remains unanswered.

EXAMPLE 2.7.4. An infinite extremely left amenable semigroup S with left cancellation, which is not uniformly left thick. As in Example 2.7.1, S is strongly left thick, but in this case S does contain finite left thick subsets.

Let S be any infinite set with the multiplication $st = t$ for all $s,t \in S$. For each $s \in S$, the set $\{s\} = Ss$ is left thick, so S is not uniformly left thick. S is extremely left amenable since for each $s \in S$ we can define $\mu_s \in M\ell(S)$ by $\mu_s(f) = f(s)$ for all $f \in m(S)$. Then each μ_s is also multiplicative.

II.8 Limitations of our techniques in determining $\dim \langle M\ell(S)\rangle$ exactly

If S is a uniformly left thick infinite left amenable semigroup, then the existence of a collection $\{D_i : i \in I\}$ of left thick subsets of S with $|D_i \cap D_j| < |S|$ for $i \neq j$, would imply $\dim \langle M\ell(S)\rangle \geq |I|$. This follows from Proposition 2.3.1, since by choosing $\mu_i \in M\ell(S)$ with $\mu_i(\chi_{D_i}) = 1$ for each $i \in I$, we have $\mu_i(\chi_{D_j}) = \mu_i(\chi_{(D_i \cap D_j)}) = 0$ when $i \neq j$. Returning to the proof of Theorem 2.2.2, we see that such a collection can be found if and only if a collection $\{P_i : i \in I\}$ of cofinal subsets of $\Gamma = \{\alpha : \alpha < \omega\}$ can be found, such that $|P_i \cap P_j| < |\Gamma|$ if $i \neq j$. When S (and hence Γ) is countably infinite, such a collection can be found with $|I| = 2^{\aleph_0} = \mathfrak{c}$ (see for instance Chou [3, p. 781]).

This leads one to hope that when Γ is infinite, one could always find such a collection with $|I| = 2^{|\Gamma|}$. However, when ω is the first uncountable ordinal, the existence of such a collection is independent of the usual axioms of set theory.[2] Thus when S is uncountable, this technique cannot be used to obtain a better lower bound for $\dim \langle M\ell(S)\rangle$.

A further example of the limitations of using collections of left thick subsets to determine the exact size of $\dim \langle M\ell(S)\rangle$ is seen by considering $\dim \langle M\ell(G)\rangle$, where G is an infinite amenable group. Using a technique of Kakutani and Oxtoby (see [21, pp. 215-225]), Chou has shown that $\dim \langle M\ell(G)\rangle = 2^{2^{|G|}}$ in [5]. Since G contains only $2^{|G|}$ distinct subsets, we cannot possibly obtain this result through the use of collections of left thick subsets.

CHAPTER III

SORENSON'S CONJECTURE AND THE STRONG FØLNER CONDITION

III.1 Introduction

The main objective of this chapter is to answer two questions. The first is Sorenson's conjecture on whether every right cancellative left amenable semigroup is left cancellative. The second, due to Argabright and Wilde, is whether every left amenable semigroup satisfies the strong Følner condition (SFC). We begin by showing that these questions are equivalent, through characterizing the semigroups which satisfy SFC as those left amenable semigroups whose right cancellative quotient semigroups are left cancellative. Then by investigating the semidirect product of semigroups in relation to amenability and cancellation properties, we are able to construct a counterexample to Sorenson's conjecture, which also shows that the answer to the question of Argabright and Wilde is no.

Sorenson's conjecture that every right cancellative left amenable semigroup is left cancellative arose as a question of John Sorenson, who proved the weaker result that every right cancellative left measurable (definition in III.5) semigroup is left cancellative in his thesis [29, p. 57] (see also [28]). The first discussion of this conjecture is found in a paper of Granirer [17, p. 108].

If this conjecture were true, then for any left amenable semigroup S, its right cancellative quotient semigroup S' (see Section I.3 for definition) would actually be cancellative and left amenable, and hence could be imbedded in an amenable group (Wilde and

Witz [33, Cor. 3.6]). Thus in some sense the study of left amenable semigroups would essentially depend on the study of left amenable subsemigroups of groups. Further interest in the conjecture arose from the work of Argabright and Wilde on the strong Følner condition.

In [11] Følner introduced the following necessary and sufficient condition for a group S to be left amenable:

(FC) For each finite subset F of S and $\varepsilon > 0$, there exists a finite subset A of S such that $|sA \setminus A| < \varepsilon|A|$ for each $s \in F$.

In his thesis [12] Frey showed that every left amenable semigroup satisfies FC; however the converse is false since every finite semigroup satisfies FC, though not every finite semigroup is left amenable. A much simpler proof of Frey's result was given by Namioka [26] using the concept of strong amenability (see Day [8, §5]).

Continuing the search for a necessary and sufficient condition of this type for left amenability in semigroups, Argabright and Wilde [1] introduced the strong Følner condition (SFC) and showed that any semigroup satisfying SFC is left amenable.

(SFC) For each finite subset F of S and $\varepsilon > 0$, there exists a finite subset A of S such that $|A \setminus sA| < \varepsilon|A|$ for each $s \in F$.

Argabright and Wilde also showed that if Sorenson's conjecture were true, then every left amenable semigroup would satisfy SFC. However, the question of whether every left amenable semigroup must satisfy SFC remained open. We will refer to this question as the SFC problem. Further discussion on this problem and Sorenson's

conjecture in relation to two conjectures of Granirer on extremely right amenable semigroups is found in Rajagopalan and Ramakrishnan [27].

In Section III.2 we show that the SFC problem and Sorenson's conjecture are equivalent, in other words every left amenable semigroup satisfies SFC if and only if every right cancellative left amenable semigroup is left cancellative. This result follows directly from Theorem 3.2.2 which completely characterizes the semigroups which satisfy SFC as those left amenable semigroups whose right cancellative quotient semigroups are left cancellative.

A counterexample to Sorenson's conjecture is constructed in Section III.3 (3.3.5). In fact we exhibit a right cancellative amenable semigroup which neither is left cancellative, nor satisfies SFC. This shows that the answer to both Sorenson's conjecture and the SFC problem is still no, even if we replace left amenable by amenable. However, since all the counterexamples we have been able to find by our method are infinitely generated, the question is still open for finitely generated semigroups. The counterexample is obtained via an investigation of the semidirect product of semigroups in relation to amenability and cancellation properties. Several other examples and results on this topic are included in Section III.3.

In Section III.4, some properties of the class of semigroups satisfying SFC are described, following the work of Day ([7] and [8]) on left amenable semigroups.

Section III.5 describes related results by Sorenson on left measurable semigroups, and concludes with a glance at semidirect products of left measurable semigroups.

III.2 Equivalence of the strong Følner condition problem to Sorenson's conjecture

After a simple lemma, we give a complete characterization of semigroups which satisfy SFC in Theorem 3.2.2. One direction of this theorem was proved by Argabright and Wilde [1]. From this characterization it will be obvious that Sorenson's conjecture and the SFC problem are equivalent (Corollary 3.2.3). Further results on the class of semigroups satisfying SFC are found in Section III.4.

LEMMA 3.2.1. Let S' be the right cancellative quotient semigroup of a semigroup S. If S' is not left cancellative then there exist $r,s,t \in S$ with $rs = rt$ but $sx \neq tx$ for each $x \in S$.

Proof. Since S' is not left cancellative, there exist $r,s_o,t_o \in S$ with $rs_oy = rt_oy$ for some $y \in S$, but $s_ox \neq t_ox$ for each $x \in S$. Now let $s = s_oy$ and $t = t_oy$.

THEOREM 3.2.2. A semigroup S satisfies SFC if and only if S is left amenable and its right cancellative quotient semigroup S' is left cancellative.

Proof. Suppose S is left amenable and S' is left cancellative. Then S' is left amenable since it is a homomorphic image of S, and hence must satisfy FC (Frey [12] or Namioka [26, Thm. 3.5]). Clearly for left cancellative semigroups, the conditions FC and SFC are equivalent, thus S' satisfies SFC. Argabright and Wilde showed that this implies that S also satisfies SFC [1, Thm. 5].

This direction of the proof is contained in Argabright and Wilde [1].

Now suppose S satisfies SFC. Then S is left amenable [1, Thm. 1], so assume S' is not left cancellative. By Lemma 3.2.1 there exist $r,s,t \in S$ with $rs = rt$ but $sx \neq tx$ for each $x \in S$. By SFC we can find a finite subset $A \subset S$ such that $|A \setminus rA| < \frac{1}{5}|A|$, $|A \setminus sA| < \frac{1}{5}|A|$, and $|A \setminus tA| < \frac{1}{5}|A|$. Now $|A \cap s^{-1}A| \geq |s(A \cap s^{-1}A)| = |sA \cap A| > \frac{4}{5}|A|$ since $|A \setminus sA| < \frac{1}{5}|A|$, and similarly $|A \cap t^{-1}A| > \frac{4}{5}|A|$. Thus $|A \cap s^{-1}A \cap t^{-1}A| > \frac{3}{5}|A|$ and hence $|A \setminus (s^{-1}A \cap t^{-1}A)| < \frac{2}{5}|A|$. This implies that $|s(A \setminus (s^{-1}A \cap t^{-1}A))| < \frac{2}{5}|A|$, and since $|sA| > \frac{4}{5}|A|$ we must have $|A \cap s(t^{-1}A)| = |s(s^{-1}A \cap t^{-1}A)| \geq |s(A \cap s^{-1}A \cap t^{-1}A)| > \frac{2}{5}|A|$.

Let $B = (A \cap s(t^{-1}A)) \cup (A \cap t(s^{-1}A))$. Clearly $|B| > \frac{2}{5}|A|$. We have $B \subset A$, and for each $y \in B$ there exists $y_o \in B \setminus \{y\}$ with $ry = ry_o$. To see this suppose $y \in A \cap s(t^{-1}A)$. Then $y = sx$ for some $x \in S$, where $tx \in A \cap t(s^{-1}A)$. Let $y_o = tx$. Clearly $y_o \in B$, $y_o \neq y$, and $ry = ry_o$ by our choice of r,s, and t. A similar argument applies for $y \in A \cap t(s^{-1}A)$. Thus we must have $|rB| \leq \frac{1}{2}|B|$.

Now we see that $|rA \cap A| \leq |rA| \leq |r(A \setminus B)| + |rB| \leq |A \setminus B| + \frac{1}{2}|B| = |A| - \frac{1}{2}|B| < \frac{4}{5}|A|$ since $|B| > \frac{2}{5}|A|$. This shows that $|A \setminus rA| > \frac{1}{5}|A|$, but A was chosen so that $|A \setminus rA| < \frac{1}{5}|A|$. Thus S' must be left cancellative.

COROLLARY 3.2.3. Every left amenable semigroup satisfies SFC if and only if every right cancellative left amenable semigroup is left cancellative.

Proof. This follows immediately from the theorem above, by noting that if S is right cancellative and left amenable, then $S = S'$.

III.3. The counterexample, semidirect products and amenability

For any semigroup U we let $\mathrm{End}(U)$ denote the set of endomorphisms of U. We use the notation $\mathrm{Inj}(U)$, $\mathrm{Sur}(U)$, and $\mathrm{Aut}(U)$ to denote the subsets of $\mathrm{End}(U)$ consisting respectively of injective endomorphisms, surjective endomorphisms, and automorphisms.

Suppose that U and T are semigroups with a homomorphism $\rho: T \to \mathrm{End}(U)$. In general we will write ρ_a for the endomorphism $\rho(a)$ for each $a \in T$. We define the <u>semidirect product of U by T</u> (with respect to ρ) as the semigroup S of ordered pairs (u,a) for $u \in U$ and $a \in T$, with the operation $(u,a)(v,b) = (u\rho_a(v),ab)$. It is easy to check that this operation is associative, hence S is indeed a semigroup. We write $S = U \times_\rho T$, and refer to U and T as the <u>factor semigroups</u>.

This product is a natural generalization of the usual semidirect product of groups (see Gorenstein [13] for example). Its extension to semigroups has already been considered from various aspects (Hofmann and Mostert [23, D.4.1], Wells [31], among others), although not in the context of amenability as far as we know.

The counterexample to Sorenson's conjecture is constructed by taking the semidirect product of two cancellative amenable semigroups in such a way that the semidirect product is right cancellative, left amenable, but not left cancellative. In Lemmas 3.3.1 and 3.3.2, and Proposition 3.3.4 we assemble the information needed to show that the example given in 3.3.5 actually has the desired properties.

The rest of this section contains other results and examples

which examine how amenability of the semidirect product is related to amenability of the factor semigroups.

LEMMA 3.3.1. If U and T are semigroups with a homomorphism $\rho: T \to \mathrm{End}(U)$ such that $\rho(T) \not\subset \mathrm{Inj}(U)$, then $S = U \underset{\rho}{\times} T$ is not left cancellative.

Proof. Suppose $a \in T$ and $u,v \in U$ with $u \neq v$ such that $\rho_a(u) = \rho_a(v)$. Then $(u,a) \neq (v,a)$ but $(u,a)(u,a) = (u,a)(v,a)$.

LEMMA 3.3.2. If U and T are right cancellative semigroups with a homomorphism $\rho: T \to \mathrm{End}(U)$, then $S = U \underset{\rho}{\times} T$ is right cancellative.

Proof. Suppose there exist $a,b,c \in T$ and $u,v,w \in U$ such that $(u,a)(w,c) = (v,b)(w,c)$. Then $ac = bc$ implies $a = b$, and $u\rho_a(w) = u\rho_b(w) = v\rho_b(w)$ implies $u = v$. Thus $(u,a) = (v,b)$.

Given a homomorphism $\rho: T \to \mathrm{End}(U)$, for each $a \in T$ we define a linear operator P_a on $m(U)$ by $P_a g(u) = g(\rho_a(u))$ for $g \in m(U)$ and $u \in U$. Each P_a induces a linear operator P_a^* on $m(U)^*$ given by $P_a^*\psi(g) = \psi(P_a g)$ for $\psi \in m(U)^*$ and $g \in m(U)$.

LEMMA 3.3.3. If U and T are left amenable semigroups with a homomorphism $\rho: T \to \mathrm{Sur}(U)$, then there exists $\phi \in M\ell(U)$ such that $P_a^*\phi = \phi$ for each $a \in T$.

Proof. For each $\psi \in M\ell(U)$ and $a \in T$ we have $P_a^*\psi \in M\ell(U)$ since ρ_a is a homomorphism of U onto U (this follows from the proof that a homomorphic image of a left amenable semigroup is also

left amenable, given in Day [7, p. 515]). Moreover, since $\rho\colon T \to \mathrm{Sur}(U)$ is a homomorphism, the map $a \to P_a^*$ is a representation of T in the set of linear mappings on $M\ell(U)$. Since $M\ell(U)$ is compact and convex in the weak*-topology and since T is left amenable, by the fixed point theorem (Day [8, Thm. 6.1]) there exists $\phi \in M\ell(U)$ with $P_a^*\phi = \phi$ for each $a \in T$.

PROPOSITION 3.3.4. If U and T are left amenable semigroups with a homomorphism $\rho\colon T \to \mathrm{Sur}(U)$, then $S = U \underset{\rho}{\times} T$ is left amenable.

Proof. By the lemma above we can choose $\phi \in M\ell(U)$ such that $P_a^*\phi = \phi$ for each $a \in T$. For each $f \in m(S)$ define $\bar{f} \in m(T)$ by $\bar{f}(a) = \phi(f_a)$, where $f_a \in m(U)$ is defined as $f_a(u) = f(u,a)$. Choose $\nu \in M\ell(T)$ and define $\mu \in m(S)^*$ by $\mu(f) = \nu(\bar{f})$. It is easy to see that μ is a mean, and moreover we claim that μ is left invariant. For $(v,b) \in S$ and $a \in T$ we have $(\ell_{(v,b)}f)_a = P_b\ell_v f_{ba}$ since for any $u \in U$, $(\ell_{(v,b)}f)_a(u) = \ell_{(v,b)}f(u,a) = f(v\rho_b(u),ba) = f_{ba}(v\rho_b(u)) = P_b\ell_v f_{ba}(u)$. Thus $\overline{(\ell_{(v,b)}f)}(a) = \phi(P_b\ell_v f_{ba}) = \phi(f_{ba}) = \ell_b\bar{f}(a)$ since $P_b^*\phi = \phi$ and ϕ is left invariant on U. Hence $\mu(\ell_{(v,b)}f) = \nu(\overline{\ell_{(v,b)}f}) = \nu(\ell_b\bar{f}) = \nu(\bar{f}) = \mu(f)$ since ν is left invariant on T. Thus $\mu \in M\ell(S)$, showing that S is left amenable.

THE COUNTEREXAMPLE 3.3.5. Let U be the free abelian semigroup generated by the elements $\{u_i \mid i = 0,1,2,\ldots\}$, and let T be the infinite cyclic semigroup with generator $\{a\}$. We define $\rho\colon T \to \mathrm{Sur}(U)$ by $\rho_a(u_i) = u_{i-1}$ if $i \geq 1$ and $\rho_a(u_o) = u_o$. Since U and T are cancellative abelian semigroups, by Lemma 3.3.2

and Proposition 3.3.4, the semigroup $S = U \times_\rho T$ is right cancellative and left amenable. However, since $\rho_a(u_1) = u_o = \rho_a(u_o)$ we have $\rho(T) \not\subset \mathrm{Inj}(U)$, and hence S is not left cancellative.

Thus S is indeed a counterexample to Sorenson's conjecture, and by Theorem 3.2.2, S is also a left amenable semigroup which does not satisfy SFC. Corollary 3.3.11 will show that S is actually amenable since U and T are amenable, which shows that Sorenson's conjecture is still false when left amenable is replaced by amenable. We have not been able to construct a finitely generated counterexample by the method above, which raises the question of whether Sorenson's conjecture holds for finitely generated semigroups.

REMARK 3.3.6. We give three examples of semidirect products of left amenable semigroups to illustrate the role that the condition $\rho: T \to \mathrm{Sur}(U)$ plays in Proposition 3.3.4. The first example shows that the condition is not necessary to ensure left amenability of $U \times_\rho T$, but examples (ii) and (iii) show that some condition is needed since neither $\rho: T \to \mathrm{End}(U)$ nor $\rho: T \to \mathrm{Inj}(U)$ is sufficient.

(i) Let U be any semigroup with at least two elements, including a zero element 0, and let T be the trivial semigroup $\{1\}$. Define $\rho_1 \in \mathrm{End}(U)$ by $\rho_1(u) = 0$ for each $u \in U$. Then for any $u,v \in U$ we have $(u,1)(v,1) = (0,1)$, thus $U \times_\rho T$ is left amenable, but since U has at least two elements $\rho_1 \notin \mathrm{Sur}(U)$.

(ii) Let T be any amenable semigroup, and U any amenable semigroup of at least two elements and containing an identity e. We define $\rho: T \to \mathrm{End}(U)$ by $\rho_a(u) = e$ for each $a \in T$ and $u \in U$. We have $(u,a)(v,b) = (ue,ab) = (u,ab)$ for any $(u,a),(v,b) \in U \times_\rho T$.

Thus if $u,v \in U$ with $u \neq v$, we see that $(u,a)(U \underset{\rho}{\times} T) \cap (v,a)(U \underset{\rho}{\times} T) = \phi$, which shows that $U \underset{\rho}{\times} T$ is not left amenable.

(iii) Let U be the non-negative integers under addition, and let T be the infinite cyclic semigroup with generator $\{a\}$. Define $\rho: T \to \mathrm{Inj}(U)$ by $\rho_a(u) = 2u$ for each $u \in U$. Now we see that

$$(0,a)(U \underset{\rho}{\times} T) = \{(u,a^j) \mid u \text{ even}, \ j = 2,3,\dots\} \text{ and}$$

$$(1,a)(U \underset{\rho}{\times} T) = \{(u,a^j) \mid u \text{ odd}, \ j = 2,3,\dots\} .$$ Thus

$(0,a)(U \underset{\rho}{\times} T) \cap (1,a)(U \underset{\rho}{\times} T) = \phi$ which shows that $U \underset{\rho}{\times} T$ is not left amenable.

REMARK 3.3.7. Suppose $S = U \underset{\rho}{\times} T$. Then we may add a two-sided identity to either U or T (or both) and extend the homomorphism ρ in such a way that S contains a two-sided ideal of the new semi-direct product obtained. Let U^o be the semigroup obtained by adding a two-sided identity e to U . Then by defining $\rho^o: T \to \mathrm{End}(U^o)$ by $\rho_a^o(u) = \rho_a(u)$ for $u \in U$ and $\rho_a^o(e) = e$ for each $a \in T$, we see that $(U^o \underset{\rho^o}{\times} T)(u,a)(U^o \underset{\rho^o}{\times} T) \subset S$ for each $(u,a) \in S$. Similarly if T^o is the semigroup obtained by adding a two-sided identity 1 to T, we define $\rho_a^o = \rho_a$ for each $a \in T$ and $\rho_1^o =$ identity homomorphism on U . Once again we have $(U \underset{\rho^o}{\times} T^o)(u,a)(U \underset{\rho^o}{\times} T^o) \subset S$ for each $(u,a) \in S$.

This remark will be useful in the propositions which follow, since it is well-known (see Day [8, p. 12 (3L‴)] or Mitchell [25, Thm. 9]) that if A is a subsemigroup of B containing a two-sided ideal of B, then A is left [right] amenable if and only if B is left [right] amenable.

PROPOSITION 3.3.8. If $S = U \underset{\rho}{\times} T$ is left amenable, then U and T are left amenable.

Proof. The map $\sigma: S \to T$ defined by $\sigma(u,a) = a$ is a homomorphism from S onto T, which shows that T is left amenable.

To show that U is left amenable, by Remark 3.3.7 we may assume without loss of generality that T has an identity 1 and that ρ_1 is the identity map on U. For each $f \in m(U)$ define $f^\sim \in m(S)$ by $f^\sim(u,a) = f(u)$. Notice that for each $v \in U$ we have $(\ell_v f)^\sim = \ell_{(v,1)} f^\sim$ since $(\ell_v f)^\sim(u,a) = \ell_v f(u) = f(vu) = f^\sim(vu,a) = \ell_{(v,1)} f^\sim(u,a)$. Choosing $\nu \in M\ell(S)$, we define $\mu \in m(U)^*$ by $\mu(f) = \nu(f^\sim)$. Actually $\mu \in M\ell(U)$ since μ is clearly a mean, and $\mu(\ell_v f) = \nu((\ell_v f)^\sim) = \nu(\ell_{(v,1)} f^\sim) = \nu(f^\sim) = \mu(f)$.

REMARK 3.3.9. The left invariant mean μ on U constructed in the above proof has the property $P_a^* \mu = \mu$ for each $a \in T$, that is $\mu(P_a f) = \mu(f)$ for each $f \in m(U)$, where $P_a f(u) = f(\rho_a(u))$. To see this, choose any $u \in U$ and notice that for $f \in m(U)$ we have $(\ell_u P_a f)^\sim = \ell_{(\rho_a(u),a)} f^\sim$, since if $(v,b) \in U \underset{\rho}{\times} T$ we have $(\ell_u P_a f)^\sim(v,b) = \ell_u P_a f(v) = P_a f(uv) = f(\rho_a(u)\rho_a(v)) = f^\sim(\rho_a(u)\rho_a(v),ab) = \ell_{(\rho_a(u),a)} f^\sim(v,b)$. Thus $\mu(P_a f) = \mu(\ell_u P_a f) = \nu((\ell_u P_a f)^\sim) = \nu(\ell_{(\rho_a(u),a)} f^\sim) = \nu(f^\sim) = \mu(f)$.

PROPOSITION 3.3.10. If U and T are right amenable semigroups with a homomorphism $\rho: T \to \mathrm{End}(U)$, then $S = U \underset{\rho}{\times} T$ is right amenable.

Proof. Choose $\phi \in Mr(U)$ and $\nu \in Mr(T)$. For each $f \in m(S)$

we define $\bar{f} \in m(T)$ by the formula $\bar{f}(a) = \phi(f_a)$, where $f_a \in m(U)$ is defined by $f_a(u) = f(u,a)$ for each $a \in T$ and $u \in U$. Now we define $\mu \in m(S)^*$ by $\mu(f) = \nu(\bar{f})$ for each $f \in m(S)$. It is easy to check that for $(v,b) \in S$ we have $(r_{(v,b)}f)_a = r_{\rho_a(v)}f_{ab}$, and hence $\overline{r_{(v,b)}f} = r_b\bar{f}$. Thus $\mu(r_{(v,b)}f) = \nu(\overline{r_{(v,b)}f}) = \nu(r_b\bar{f}) = \mu(f)$. Since μ is also a mean on S, we see that S is right amenable.

COROLLARY 3.3.11. If U and T are amenable semigroups with a homomorphism $\rho: T \to \mathrm{Sur}(U)$, then $S = U \underset{\rho}{\times} T$ is amenable.

Proof. This is immediate from Propositions 3.3.4 and 3.3.10.

PROPOSITION 3.3.12. If $S = U \underset{\rho}{\times} T$ is right amenable then T is right amenable, and if in addition $\rho: T \to \mathrm{Aut}(U)$ then U is right amenable also.

Proof. As before T is a homomorphic image of S and hence right amenable. Now suppose $\rho: T \to \mathrm{Aut}(U)$. By Remark 3.3.7 we may assume that T has an identity 1 and that ρ_1 is the identity map on U. For $f \in m(U)$ we define $f^\vee \in m(S)$ by $f^\vee(u,a) = f(\rho_a^{-1}(u))$, where ρ_a^{-1} denotes the inverse automorphism to ρ_a. Now for each $v \in U$, we have $(r_v f)^\vee(u,a) = f(\rho_a^{-1}(u)v) = f(\rho_a^{-1}(u\rho_a(v))) = f^\vee(u\rho_a(v),a) = r_{(v,1)}f^\vee(u,a)$. Choosing $\nu \in Mr(S)$ we define $\mu \in m(U)^*$ by $\mu(f) = \nu(f^\vee)$. It is easy to see that μ is a right invariant mean on U.

COROLLARY 3.3.13. If $S = U \underset{\rho}{\times} T$ is amenable then T is amenable, and if in addition $\rho: T \to \mathrm{Aut}(U)$, then U is also amenable.

Proof. This is immediate from Propositions 3.3.8 and 3.3.12.

REMARK 3.3.14. We give two examples to show that the condition $\rho: T \to \mathrm{Aut}(U)$ in Proposition 3.3.12 cannot be replaced by either $\rho: T \to \mathrm{Sur}(U)$ or $\rho: T \to \mathrm{Inj}(U)$. Example (iii) shows that we cannot replace $\rho: T \to \mathrm{Aut}(U)$ in Corollary 3.3.13 by $\rho: T \to \mathrm{End}(U)$.

(i) We construct a right amenable semigroup $S = U \underset{\rho}{\times} T$ where $\rho: T \to \mathrm{Sur}(U)$, but U is not right amenable. Let U be the free semigroup on the generators $\{u_i | i = 0,1,2,\ldots\}$, and let T be the infinite cyclic semigroup generated by $\{a\}$. We define $\rho: T \to \mathrm{Sur}(U)$ by $\rho_a(u_i) = u_{i-1}$ for $i \geq 1$ and $\rho_a(u_o) = u_o$. Since U is clearly not right amenable, all that remains to be shown is that S is right amenable. Actually we show that S satisfies the "right-sided" version of the strong Følner condition: (SFC_r) For any finite subset $F \subset S$ and $\varepsilon > 0$, there exists a finite subset $A \subset S$ with $|A \setminus As| < \varepsilon|A|$ for each $s \in F$.

Thus suppose F is a finite subset and $\varepsilon > 0$. Then there exists an integer N such that

$$\{(\rho_{a^N}(u), a^{\ell}) | (u, a^{\ell}) \in F\} \subset \{(u_o^j, a^k) | 1 \leq j,k \leq N\} .$$

Choose M so that $2N/M < \varepsilon$, and let $A = \{(u_o^m, a^{n+N}) | 1 \leq m,n \leq M\}$. Then for each $(u, a^{\ell}) \in F$ it is easy to check that

$$\{(u_o^m, a^{n+N}) | N + 1 \leq m,n \leq M\} \subset A(u, a^{\ell}),$$

which shows that $|A \setminus A(u, a^{\ell})| < 2NM = (2N/M)M^2 < \varepsilon|A|$.

(ii) We construct a right amenable semigroup $S = U \underset{\rho}{\times} T$ where

$\rho: T \to \mathrm{Inj}(U)$, but U is not right amenable. Let U be the semigroup generated by the elements $\{u_i, v_i | i = 1,2,\ldots\}$ with the relations $u_i v_j = v_j u_i = u_i u_j = u_j u_i = u_i$ if $i < j$, and

$u_i v_j = v_j u_i = v_i v_j = v_j v_i = v_j$ if $j < i$. Notice that for each i, the semigroup generated by $\{u_i, v_i\}$ is the free semigroup on two generators, and hence $Uu_1 \cap Uv_1 = \phi$. This shows that U is not right amenable.

Let T be the infinite cyclic semigroup generated by $\{a\}$, and define $\rho: T \to \mathrm{Inj}(U)$ by $\rho_a(u_i) = u_{i+1}$ and $\rho_a(v_i) = v_{i+1}$. To see that $S = U \underset{\rho}{\times} T$ is right amenable we show that S satisfies SFC_r as in example (i) above. Suppose $F \subset S$ is a finite subset and $\varepsilon > 0$. Let $N = \sup\{n | (u,a^n) \in F$ for some $u \in U\}$, and choose M so that $N/M < \varepsilon$. Letting $A = \{(u_1, a^k) | 1 \le k \le M\}$, we see that for each $(u,a^n) \in F$ we have $A(u,a^n) = \{(u_1, a^{k+n}) | 1 \le k \le M\}$ since $u_1 \rho_a(u) = u_1$ for all $u \in U$. Also since $n \le N$ we have $|A \setminus A(u,a^n)| \le N = (N/M)|A| < \varepsilon |A|$.

(iii) We construct an amenable semigroup $S = U \underset{\rho}{\times} T$ with $\rho: T \to \mathrm{End}(U)$, such that U is not right amenable, hence not amenable. Let $U = \{u,v\}$ where $u^2 = vu = u$ and $v^2 = uv = v$. We choose T to be the trivial semigroup $\{1\}$, and define $\rho: T \to \mathrm{End}(U)$ by $\rho_1(u) = \rho_1(v) = u$. Clearly U is not right amenable since $Uu \cap Uv = \phi$. However $(u,1)(v,1) = (v,1)(u,1)$ which shows that S is abelian and hence amenable.

III.4 Semigroups satisfying the strong Følner condition

From the counterexample constructed in 3.3.5 we know that the class of semigroups satisfying SFC is a proper subset of the class of left amenable semigroups. In this section we examine some of the properties of this class, generally following the line of results established for left amenable semigroups (see Day [7] and [8]).

REMARK 3.4.1. By Lemma 3.2.1 and Theorem 3.2.2, in order to show that a particular left amenable semigroup S satisfies SFC, we need only show that whenever $rs = rt$ for some $r,s,t \in S$, there exists $x \in S$ with $sx = tx$.

PROPOSITION 3.4.2. If S is a semigroup satisfying SFC with a subsemigroup T such that $\mu(\chi_T) > 0$ for some $\mu \in M\ell(S)$, then T also satisfies SFC.

Proof. T is left amenable since $\mu(\chi_T) > 0$ (Day [7]). Suppose $a,b,c \in T$ with $ab = ac$. Since S satisfies SFC there exists $x \in S$ with $bx = cx$. Since $\mu(\chi_T) > 0$ and $\mu(\chi_{xS}) = 1$, $T \cap xS$ is non-empty. Hence we can pick $d \in T \cap xS$, and we have $bd = cd$. This shows that T satisfies SFC by Remark 3.4.1.

It is not true, however, that every left amenable subsemigroup of a semigroup satisfying SFC must satisfy SFC. For example, consider a semigroup obtained by adding a two-sided zero to another semigroup which is left amenable but does not satisfy SFC.

Recall from Section I.4 that a subset T of S is said to be left thick in S if for every finite subset $F \subset S$, there exists

$s \in S$ such that $Fs \subset T$.

PROPOSITION 3.4.3. If T is a left thick subset of S such that T satisfies SFC, then S satisfies SFC.

Proof. Let F be a finite subset of S and $\varepsilon > 0$. Choose $\delta > 0$ such that $\frac{2\delta}{1-\delta} < \varepsilon$. Since T is left thick there exists $a \in T$ such that $Fa \subset T$. Choose a finite subset $B \subset T$ such that $|B \setminus aB| < \delta|B|$ and $|B \setminus saB| < \delta|B|$ for each $s \in F$. Now we see that $|aB \setminus saB| < 2\delta|B| < 2\delta\left(\frac{|aB|}{(1-\delta)|B|}\right)|B| < \varepsilon|aB|$.

This proposition extends a result of Rajagopalan and Ramakrishnan [27, Thm. 22]. The next two results are stated without proof since their verification is routine.

PROPOSITION 3.4.4. A finite direct product of semigroups which satisfy SFC also satisfies SFC .

PROPOSITION 3.4.5. A directed union of semigroups which satisfy SFC also satisfies SFC.

PROPOSITION 3.4.6. If U and T are semigroups which satisfy SFC with a homomorphism $\rho: T \to \mathrm{Aut}(U)$, then $S = U \underset{\rho}{\times} T$ satisfies SFC.

Proof. S is left amenable by Proposition 3.3.4. Suppose there exist (u,a), (v,b), and $(w,c) \in S = U \underset{\rho}{\times} T$ with $(u,a)(v,b) = (u,a)(w,c)$. Then $u\rho_a(v) = u\rho_a(w)$ implies $\rho_a^{-1}(u)v = \rho_a^{-1}(u)w$, and hence there exists $y \in U$ with $vy = wy$. Also $ab = ac$ implies there exists $d \in T$ with $bd = cd$. Since ρ_d is surjective and

$\rho_b\rho_d = \rho_c\rho_d$ we must have $\rho_b = \rho_c$. Thus we see that $(v,b)(\rho_b^{-1}(y),d) = (vy,bd) = (wy,cd) = (w,c)(\rho_c^{-1}(y),d) = (w,c)(\rho_b^{-1}(y),d)$. By Remark 3.4.1 we have shown that S satisfies SFC.

Examples 3.3.5 and 3.3.6 (iii) show respectively that the condition $\rho\colon T \to \mathrm{Aut}(U)$ in the proposition above cannot be replaced by either $\rho\colon T \to \mathrm{Sur}(U)$ or $\rho\colon T \to \mathrm{Inj}(U)$.

PROPOSITION 3.4.7. If $S = U \underset{\rho}{\times} T$ satisfies SFC, then U and T also satisfy SFC.

Proof. U and T are left amenable by Proposition 3.3.8. Moreover, Remark 3.3.7 combined with Propositions 3.4.2 and 3.4.3 shows that we may assume that T has an identity 1 and that ρ_1 is the identity map on U . Suppose $uv = uw$ for $u,v,w \in U$. Then $(u,1)(v,1) = (u,1)(w,1)$, hence there exists $(y,a) \in S$ with $(v,1)(y,a) = (w,1)(y,a)$. From this we see $vy = wy$, which shows that U satisfies SFC.

Now suppose $ab = ac$ for $a,b,c \in T$. Then $(u,a)(u,b) = (u,a)(u,c)$ for any $u \in U$, thus there exists $(v,d) \in S$ with $(u,b)(v,d) = (u,c)(v,d)$. Hence we have $bd = cd$, showing that T satisfies SFC.

The question of whether homomorphic images of semigroups satisfying SFC also satisfy SFC is still open. An answer to this would be particularly interesting in view of the corresponding results for left amenable semigroups (yes, Day [7]) and left measurable semigroups (no, Sorenson [29]).

III.5 Left measurable semigroups

For any semigroup S it is easy to see that a mean $\mu \in m(S)^*$ is left invariant if and only if $\mu(\chi_{s^{-1}A}) = \mu(\chi_A)$ for each $A \subset S$ and $s \in S$. We say that a mean $\mu \in m(S)^*$ is <u>left reversible invariant</u> if $\mu(\chi_{sA}) = \mu(\chi_A)$ for each $A \subset S$ and $s \in S$, and denote the set of left reversible invariant means on S by $RM\ell(S)$. If a semigroup S has a left reversible invariant mean we say that S is <u>left measurable</u>. This term arises from the obvious one-to-one correspondence between $RM\ell(S)$ and the set of left measures on S, i.e. the set of finitely additive measures λ on S such that $\lambda(S) = 1$ and $\lambda(sA) = \lambda(A)$ for each $s \in S$ and $A \subset S$. Clearly every left measurable semigroup is left amenable since any left reversible invariant mean is left invariant, and also for left cancellative semigroups the conditions are equivalent. The terms right reversible invariant and right measurable are defined analogously.

Sorenson investigated the properties of left measurable semigroups in his thesis [29]. In particular he showed that every left measurable right cancellative semigroup is left cancellative [29, 3.1.7]. The proof that follows in Theorem 3.5.1 is not the one that Sorenson gave, although he noticed that this type of proof was possible [29, Remarks on p. 57].

THEOREM 3.5.1. If S is a left measurable semigroup, then its right cancellative quotient semigroup S' is left cancellative.

Proof. Suppose S' is not left cancellative. Then by Lemma 3.2.1 there exist $r,s,t \in S$ such that $rs = rt$ but $sx \neq tx$ for each $x \in S$. Let $\mathcal{A} = \{A \subset S \mid sA \cap tA = \phi\}$. Then $\mathcal{A} \neq \phi$ since $\{x\} \in \mathcal{A}$ for each $x \in S$. If we partially order $\mathcal{A}$ by inclusion, it is easy to see that $\mathcal{A}$ is chainable, thus by Zorn's lemma let A be a maximal element in $\mathcal{A}$. For each $x \in S \setminus A$ we have either $sx \in tA$ or $tx \in sA$ since $sx \neq tx$ and A is maximal. Thus we may write $S = A \cup S_1 \cup S_2$ where $sS_1 \subset tA$ and $tS_2 \subset sA$. If $\mu \in RM\ell(S)$, we must have $\mu(\chi_A) = \mu(\chi_{tA}) \geq \mu(\chi_{sS_1}) = \mu(\chi_{S_1})$ and similarly $\mu(\chi_A) \geq \mu(\chi_{S_2})$, which shows that $\mu(\chi_A) \geq \frac{1}{3}$. However, $rs = rt$ implies that $\mu(\chi_A) = \mu(\chi_{rsA}) = \mu(\chi_{(sA \cup tA)}) = 2\mu(\chi_A)$, and hence $\mu(\chi_A) = 0$. By this contradiction we see that S' must be left cancellative.

COROLLARY 3.5.2. If S is left measurable and right cancellative, then S is left cancellative.

Proof. If S is right cancellative then $S = S'$.

COROLLARY 3.5.3. Every left measurable semigroup satisfies SFC.

Proof. This follows immediately from Theorems 3.2.2 and 3.5.1.

It is not true that every semigroup which satisfies SFC is left measurable, since any semigroup with a zero element obviously satisfies SFC, but cannot be left measurable if it has more than one element.

We now state some of the properties of left measurable semigroups obtained by Sorenson.

(i) The homomorphic image of a left measurable semigroup is not necessarily left measurable [29, Example 1, §3.1].

(ii) A left ideal of a left measurable semigroup need not be left measurable [29, Example 2, §3.1].

(iii). A right ideal of a left measurable semigroup is left measurable [29, 3.1.2].

(iv) A finite direct product of left measurable semigroups is left measurable [29, 3.1.4].

(v) A directed union of left measurable semigroups is left measurable [29, 3.1.5].

The next lemma is useful in proving Propositions 3.5.5 and 3.5.6, which look at the semidirect product in relation to left measurability, following the pattern of results established in Sections III.3 and III.4 for amenable semigroups and semigroups satisfying SFC respectively.

LEMMA 3.5.4. A mean μ is left reversible invariant on a semigroup S if and only if μ is left invariant and $\mu(\chi_{Z_s}) = 1$ for each $s \in S$, where $Z_s = \{t \in S \mid s^{-1}\{st\} = \{t\}\}$.

Proof. Suppose $\mu \in RM\ell(S)$ and $s \in S$. It is clear from the definition of Z_s, that by using a Zorn's lemma argument we may write $S \setminus Z_s = A_1 \cup A_2$, where $A_1 \cap A_2 = \phi$ and $sA_1 = sA_2 = s(S \setminus Z_s)$. This implies that $\mu(\chi_{A_1}) = \mu(\chi_{A_2}) = \mu(\chi_{(S\setminus Z_s)})$ and also that $\mu(\chi_{(S\setminus Z_s)}) = \mu(\chi_{A_1}) + \mu(\chi_{A_2})$. From this we see that

$\mu(\chi_{(S\setminus Z_s)}) = 0$, and hence $\mu(\chi_{Z_s}) = 1$.

Now suppose that $\mu \in M\ell(S)$ and $\mu(\chi_{Z_s}) = 1$ for each $s \in S$. For any set $A \subset S$ we have $\left(s^{-1}(sA) \setminus A\right) \subset \left(S \setminus Z_s\right)$, and hence $\mu(\chi_{(s^{-1}(sA)\setminus A)}) = 0$. Thus $\mu(\chi_{sA}) = \mu(\chi_{s^{-1}(sA)}) = \mu(\chi_A) + \mu(\chi_{(s^{-1}(sA)\setminus A)}) = \mu(\chi_A)$, which shows that μ is left reversible invariant on S.

The first part of this proof is given by Sorenson in [29, 2.3.2].

Notice that for a left amenable semigroup S to be left measurable it must be "almost" left cancellative, in the sense that Z_s must be left thick in S for each $s \in S$.

PROPOSITION 3.5.5. If U and T are left measurable semigroups with a homomorphism $\rho\colon T \to \mathrm{Aut}(U)$, then $S = U \underset{\rho}{\times} T$ is left measurable.

Proof. Recall that in Section III.3 we defined $P_a\colon m(U) \to m(U)$ by $P_a f(u) = f(\rho_a(u))$, which induced a linear operator P_a^* on $m(U)^*$. Since $\rho_a \in \mathrm{Aut}(U)$, it is straightforward to show that P_a^* maps $RM\ell(U)$ onto itself, and since $RM\ell(U)$ is compact and convex with respect to the weak*-topology (Sorenson [29, 1.1.10]), once again we apply the fixed point theorem (Day [8, Thm. 6.1]) to obtain a left reversible invariant mean ϕ on U such that $P_a^*\phi = \phi$ for each $a \in T$.

For each $f \in m(S)$ define $f_a \in m(U)$ by $f_a(u) = f(u,a)$, and define $\bar{f} \in m(T)$ by $\bar{f}(a) = \phi(f_a)$. Choosing $\nu \in RM\ell(T)$ we define $\mu \in m(S)^*$ by $\mu(f) = \nu(\bar{f})$. From the proof of Proposition 3.3.4 we

know that $\mu \in M\ell(S)$, thus by Lemma 3.5.4 we need only show that $\mu(\chi_{Z_{(u,a)}}) = 1$ for each $(u,a) \in S$. After noting that $Z_{(u,a)} = \{(v,b) \mid v \in Z_{\rho_a^{-1}(u)}$ and $b \in Z_a\}$, we see that $(\chi_{Z_{(u,a)}})_b = \chi_{Z_{\rho_a^{-1}(u)}}$ if $b \in Z_a$, and $(\chi_{Z_{(u,a)}})_b = 0$ if $b \notin Z_a$. Since $\phi(\chi_{Z_{\rho_a^{-1}(u)}}) = 1$ by Lemma 3.5.4, we see that $\overline{\chi_{Z_{(u,a)}}} = \chi_{Z_a}$. Now we have $\mu(\chi_{Z_{(u,a)}}) = \nu(\chi_{Z_a}) = 1$ by Lemma 3.5.4, as desired.

It is not possible to replace the condition $\rho\colon T \to \mathrm{Aut}(U)$ in this proposition by either $\rho\colon T \to \mathrm{Sur}(U)$ or $\rho\colon T \to \mathrm{Inj}(U)$, as is shown by the examples 3.3.5 and 3.3.6 (iii) respectively.

PROPOSITION 3.5.6. If $S = U \underset{\rho}{\times} T$ is left measurable, then U and T are left measurable.

<u>Proof</u>: Let $\nu \in RM\ell(S)$. To see that U is left measurable, we define $\mu \in m(U)^*$ as in Proposition 3.8 by $\mu(f) = \nu(f^\sim)$, where $f^\sim \in m(S)$ is defined by $f^\sim(u,a) = f(u)$ for each $f \in m(U)$. In the proof of Proposition 3.3.8 we saw that $\mu \in M\ell(U)$, hence it suffices to show that $\mu(\chi_{Z_u}) = 1$ for each $u \in U$. First consider the set $Z_{(\rho_a(u),a)}$ for some fixed $a \in T$. Now we have $(\chi_{Z_u})^\sim \geq \chi_{Z_{(\rho_a(u),a)}}$, since if $v \notin Z_u$ there exists $w \neq v$ with $uv = uw$. Then for any $b \in T$ we have $(\rho_a(u),a)(v,b) = (\rho_a(u),a)(w,b)$ which shows that $(v,b) \notin Z_{(\rho_a(u),a)}$ for each $b \in T$.

Now we have $1 \geq \mu(\chi_{Z_u}) = \nu((\chi_{Z_u})^{\sim}) \geq \nu(\chi_{Z_{(\rho_a(u),a)}}) = 1$ by Lemma 3.5.4.

Similarly, to see that T is left measurable we define $\psi \in m(T)^*$ by $\psi(g) = \nu(g^\wedge)$, where $g^\wedge \in m(S)$ is defined by $g^\wedge(u,a) = g(a)$ for each $g \in m(T)$. After checking that ψ is a left invariant mean, we see that $\psi(\chi_{Z_a}) = 1$ for each $a \in T$, since $(\chi_{Z_a})^\wedge \geq \chi_{Z_{(u,a)}}$ for any $u \in U$.

Combining the appropriate version of Lemma 3.5.4 for right reversible invariant means with Propositions 3.3.10 and 3.3.12, analogous arguments yield the following results:

PROPOSITION 3.5.7. If U and T are right measurable semigroups with a homomorphism $\rho: T \to \mathrm{End}(U)$, then $S = U \times_\rho T$ is right measurable.

PROPOSITION 3.5.8. If $S = U \times_\rho T$ is right measurable and $\rho: T \to \mathrm{Aut}(U)$, then U and T are right measurable.

FOOTNOTES

1. In Granirer's original theorem [14, p. 32], the condition "S contains exactly n disjoint finite groups which are left ideals" was stated as "S contains exactly n disjoint finite left ideal left cancellative groups" (a set $A \subset S$ is said to be left ideal left cancellative if A is a left ideal in S, and whenever $sa = sb$ for $s \in S$ and $a,b \in A$, then $a = b$). However, as Granirer later noted, if G is a finite group with identity e, and also a left ideal in S, then for each $s \in S$ we have $sG = s(eG) = (se)G = G$, which implies that G is left ideal left cancellative.

2. This result was brought to my attention by R. Sirois-Dumais. It is stated in a preprint "Combinatorics" by K. Kunen. He has sent me the proof in a letter, mentioning that it was first obtained by Baumgartner in his Ph.D. thesis.

BIBLIOGRAPHY

1. L.N. Argabright and C.O. Wilde, Semigroups satisfying a strong Følner condition, Proc. Amer. Math. Soc. 18 (1967), 587-591.

2. S. Banach, Sur la problème de la mesure, Fund. Math. 4 (1923), 7-33.

3. C. Chou, Minimal sets and ergodic measures for $\beta\mathbb{N}\setminus\mathbb{N}$, Illinois J. of Math 13 (1969), 777-788.

4. C. Chou, On the size of the set of left invariant means on a semigroup, Proc. Amer. Math. Soc. 23 (1969), 199-205.

5. C. Chou, The exact cardinality of the set of invariant means on a group, Proc. Amer. Math. Soc. 55 (1976), 103-106.

6. P. Civin and B. Yood, The second conjugate space of a Banach algebra as an algebra, Pacific J. Math. 11 (1961), 847-870.

7. M.M. Day, Amenable semigroups, Illinois J. of Math. 1 (1957), 509-544.

8. M.M. Day, Semigroups and amenability, Semigroups, Ed. K. Folley, Academic Press, New York, 1969, 5-53.

9. J. Dixmier, Les moyennes invariantes dans les semigroupes, et leur applications, Acta. Sci. Math. (Szeged), 12 (1950), 213-227.

10. L.R. Fairchild, Extreme invariant means without minimal support, Trans. Amer. Math. Soc. 172 (1972), 83-93.

11. E. Følner, On groups with full Banach mean values, Math. Scand. 3 (1955), 243-254.

12. A.H. Frey, Jr., Studies on amenable semigroups, Thesis, University of Washington, 1960.

13. D. Gorenstein, Finite Groups, Harper and Row, New York, 1968.

14. E. Granirer, On amenable semigroups with a finite-dimensional set of invariant means, I, II, Illinois J. of Math. 7 (1963), 32-58.

15. E. Granirer, A theorem on amenable semigroups, Trans. Amer. Math. Soc. 111 (1964), 367-379.

16. E. Granirer, Extremely amenable semigroups, Math. Scand. 17 (1965), 177-197.

17. E. Granirer, Extremely amenable semigroups II, Math. Scand. 20 (1967), 93-113.

18. E. Granirer and M. Rajagopalan, A note on the radical of the second conjugate algebra of a semigroup algebra, Math. Scand. 15 (1964), 163-166.

19. F.P. Greenleaf, Invariant means on topological groups and their applications, Van Nostrand Math. Studies, no. 16, Van Nostrand Reinhold, New York, 1969.

20. F. Hausdorff, Grundzüge der Mengenlehre, Leipzig, 1914.

21. E. Hewitt and K. Ross, Abstract harmonic analysis, Vol. 1, Springer, Berlin, 1963.

22. M. Hochster, Subsemigroups of amenable groups, Proc. Amer. Math. Soc. 21 (1968), 363-364.

23. K.H. Hofmann and P.S. Mostert, Elements of compact semigroups, C.E. Merrill Books, Columbus, Ohio, 1966.

24. I.S. Luthar, Uniqueness of the invariant mean on an abelian semigroup, Illinois J. of Math. 3 (1959), 28-44.

25. T. Mitchell, Constant functions and left invariant means on semigroups, Trans. Amer. Math. Soc. 119 (1965), 244-261.

26. I. Namioka, Følner's conditions for amenable semi-groups, Math. Scand. 15 (1964), 18-28.

27. M. Rajagopalan and P.V. Ramakrishnan, On a conjecture of Granirer and strong Følner condition, J. of Indian Math. Soc. 37 (1973), 85-92.

28. J.R. Sorenson, Left-amenable semigroups and cancellation, Notices of A.M.S., Vol. 11, No. 7, November 1964, p. 763.

29. J.R. Sorenson, Existence of measures that are invariant under a semigroup of transformations, Thesis, Purdue University (1966).

30. J. von Neumann, Zur allgemeine Theorie des Masses, Fund. Math. 13 (1929), 73-116.

31. C. Wells, Some applications of the wreath product construction, Amer. Math. Monthly 83 (1976), 317-338.

32. C.O. Wilde and L.N. Argabright, Invariant means and factor-semigroups, Proc. Amer. Math. Soc. 18 (1967), 226-228.

33. C.O. Wilde and K. Witz, Invariant means and the Stone-Čech compactification, Pacific J. Math 21 (1967), 577-586.